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17/1/2022

MATHS407 (ADVANCED REAL ANALYSIS)

COURSE OUTLINE

1 Uniform Convergence of sequence and series of functions pointwise and uniform convergences, Cauchy's general principle of uniform convergence, test for uniform convergence. Mn-test, Weierstrass M test, Abel's test, Dirichlet's test. Uniform convergence and continuity. Dini's theorem. Integrability of uniform limit of a uniformly convergent series of integrable functions, term by term integration

2 Uniform convergence and differentiability, Weierstrass's continuous non-differentiable function. Uniform convergence of power series.

3- Functions of bounded variation and their properties:- variation function of a function of bounded variation Jordan's theorem. ~~Riemann - Stieltjes~~

4 Riemann - Stieltjes Integral:- Stieltjes Integral and its various generalizations, conditions of integrability, Integration by parts. first mean value theorem, second mean value theorem, Differentiation under the integral sign

REFERENCE

1- The elements of Real Analysis, Bartle R.G. John Willey (19..

2- Mathematical Analysis, Apostle T.M, Addison - Willey (1974).

DEFINITION 1.0: A sequence is a function whose domain is a natural number and the range is a set of real numbers i.e. a real sequence is a function $x_n : \mathbb{N} \longrightarrow \mathbb{R}$.

DEF 1.1 A sequence $x_n : \mathbb{N} \longrightarrow \mathbb{R}$ is denoted by $\{x_n\}_{n\in\mathbb{N}}$ or $\{x_n\}$, or (x_n) or $(x_1, x_2, \ldots)$ or $\{x_1, x_2, x_3 \ldots\}$. The terms $x_1, x_2, x_3 \ldots$ are called the first, second, third ... terms of the sequence.

Observe the terms of a sequence are infinite. However, the range may be finite or infinite. E.g the following are sequence $\{x_n\} = \{1/n\}$, $\{x_n\} = \{1 + 1/n\}$, $\{x_n\} = \{(-1)\}$ etc.

DEF 1.2 A sequence (x_n) is said to be bounded above if $\exists$ a real number α, $\ni x_n \leq \alpha \ \forall n$. (x_n) is said to be bounded below if $\exists$ a real number $\beta \ni x_n \geq \beta \ \forall n$. (x_n) is said to be bounded if it is bounded above and below.

If (x_n) is bounded above by α and bounded below by β. Then α and β are called the upper and lower bounds respectively of (x_n).

DEF 1.3 A sequence (x_n) is said to converge to a real number p if, $\forall \varepsilon > 0$, $\exists$ a positive integer m, $\ni |x_n - p| < \varepsilon$, $\forall n > m$.

DEF 1.4 A sequence of functions $\{f_n\}$ is a sequence whose terms are real-valued functions defined on same interval say $I = [a, b]$, Hence, for each p in I, $\{f_n\}$ corresponds to $f_1(p), f_2(p)$.....

DEF 1.5 A sequence of functions $\{f_n\}$ is said to be pointwise convergent on $[a, b]$ if for each $\varepsilon > 0$ and for each $x \in [a, b]$, $\exists$ a positive integer $m = m(\varepsilon, x)$ such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq m \quad \text{---} \quad (i)$$

The function f in (i) is called the pointwise limit of $f_n(x)$ from (i), ~~we~~ also write:

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{or} \quad f_n(x) \xrightarrow{n \to \infty} f(x)$$

$$\text{or} \quad f_n(x) \xrightarrow[n \to \infty]{} f(x) \quad \text{or} \quad f_n(x) \longrightarrow f(x) \text{ as } n \to \infty$$

The notation $m(\varepsilon, x)$ means that pointwise convergence of (f_n) to the limit function f depends on both ε and x.

DEF 1.6: A sequence of functions $\{f_n\}$ is said to converge uniformly on $[a, b]$, if for each $\varepsilon > 0$ and $\forall x \in [a, b]$ $\exists$ a positive integer $m(\varepsilon)$ depending on ε, $\ni$ $\forall\, x \in [a, b]$; then

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq m(\varepsilon) \quad \text{---} \quad (ii)$$

The function in (ii) is called the uniform limit of (f_n).

NB: In the case of point-wise convergence, for each $\varepsilon > 0$ and for each $x \in [a,b]$, $\exists$ a ~~pointwise convergen~~ ~~for each~~ positive Integer $m(\varepsilon, x)$ depending on both ε and x, such that eqn (i) holds. ~~Where~~ ~~as~~ ~~if~~ for Uniform Convergence for each $\varepsilon > 0$, it is possible to find a positive Integer $m(\varepsilon)$ depending only on ε which will suffice for all $x \in [a,b]$. It follows that every Uniform Convergence is pointwise convergence. However, not point wise Convergent Implies not uniform Convergent. This also means that pointwise limit is the same as the Uniform limit.

DEF. 17: A series of function $\sum_{n=1}^{\infty} f_n(x)$ is said to converge pointwise to a function f defined on (a,b) if its sequence of nth partial sum $\{f_n\}$ converges pointwise to f.

$|S_n - f| < \varepsilon = \left|\sum_{i=1}^{n} f_i(x) - f\right| < \varepsilon$. Similarly for Uniform Convergen

DEFINITION: A sequence $\{x_n\}$ of numbers is said to be Cauchy if for every $\varepsilon > 0$, $\exists\, \alpha \in N \ni |x_n - x_m| < \varepsilon$, $\forall\, n,m > \alpha$.

Cauchy Convergence Criterion

A sequence $\{x_n\}$ is Convergent iff it is a chauchy

Cauchy Convergence Criterion for sequence of functions.

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THEOREM: A sequence of functions $\{f_n\}$ Converges uniformly on $[a,b]$, iff for any $\epsilon > 0$, and for all $x \in [a,b]$, there exists a natural number $\lambda(\epsilon)$ such that

$$|f_n(x) - f_m(x)| < \epsilon, \quad \forall\, n,m \geq \lambda(\epsilon) \quad \text{———} \quad (*)$$

proof: Necessary ($\Rightarrow$): Suppose $\{f_n\}$ Converges uniformly on $[a,b]$ to the limit function f. This means for every $\varepsilon > 0$ and for all $x \in [a,b]$, there exist positive Integer $\eta_1(\epsilon)$, $\eta_2(\epsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \forall\, n \geq \eta_1(\epsilon) \quad \text{———} \quad (1)$$

$$|f_m(x) - f(x)| < \varepsilon/2 \quad \forall\, m \geq \eta_2(\epsilon) \quad \text{———} \quad (2)$$

Let $\lambda(\epsilon) = \max(\eta_1(\epsilon), \eta_2(\epsilon))$ then for the given $\varepsilon > 0$ and for all $x \in [a,b]$ we have

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall\, n,m \geq \lambda(\epsilon) \end{aligned}$$

sufficiently ($\Leftarrow$): Assume that $(*)$ holds.
Then by Cauchy Convergence Criterion $\{f_n\}$ Converges to f; (say) pointwise. To see that this Convergence is uniformly from $(*)$ let n be fixed, and $m \to \infty$. Then for the given $\varepsilon > 0$, and for all $x \in [a,b]$, we have

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq \lambda(\varepsilon)$$

This Implies $f_n \xrightarrow[n \to \infty]{\text{uniformly}} f$ $\quad \square$

Example 1 Test for uniform Convergence of the sequence $\{f_n\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$.

Soln first, we show that the sequence Converges pointwise for this, we see that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0 = f(x) \quad \text{that is,}$$

$$f_n(x) \xrightarrow[n\to\infty]{\text{pointwise}} f(x) = 0 \text{ on } \mathbb{R}$$

for uniform Convergence, let $\varepsilon > 0$ be given. then for all $x \in [a,b]$, we need to show that there exist a positive natural $\eta(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq \eta(\varepsilon) \quad \text{—— (1)}$$

Consider, the LHS of (1):

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} = \frac{1}{nx} < \varepsilon$$

provided $n > \frac{1}{\varepsilon x} = \eta(\varepsilon, x)$

obviously, for $x = 0 \in \mathbb{R}$, $\eta(\varepsilon, 0) = \frac{1}{0} \to \infty$

this means that we cannot find an $\eta(\varepsilon)$ that depends only on ε such that (1) is valid. Hence $\{f_n\}$ does not converges uniformly for all $x \in \mathbb{R}$.

NB: Every uniformly Convergent $\Rightarrow$ pointwise but not convergently. However, not pointwise implies not uniform.

Example 2 Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{1}{x+n}$ is uniformly convergent in an interval $[0, b]$, $b > 0$

Soln first, for pointwise limit

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0 = f(x)$$

$$\Rightarrow f_n(x) \xrightarrow[n\to\infty]{} f(x) = 0 \text{ on } [0, b]$$

for uniform convergence given any $\varepsilon > 0$ and for all $x \in [0, b]$ we need to find a number $w(\varepsilon) > 0$, $\ni$

$$|f_n(x) - f(x)| < \varepsilon, \ \forall\ n \geq w(\varepsilon) \quad \text{———} (1)$$

from the L.H.S of (1)

$$|f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right| = \frac{1}{x+n} < \frac{1}{n} < \epsilon, \text{ if } n > \frac{1}{\varepsilon} = w(\epsilon)$$

$\therefore f_n(x) \longrightarrow f(x) = 0$ uniformly as $n \to \infty$ on $[0, b]$

Example 3 Show that the sequence $\{f_n\}$, where $f_n(x) = x^n$ is uniformly convergent on $[0, c]$, $c < 1$ and only pointwise on $[0, 1]$.

Soln for pointwise, we see that

$$f(x) = \lim_{n\to\infty} f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

we see that $f_n(x)$ converges pointwise to a discontinuous function $f(x)$.

for uniformly continuity, we consider for $0 < x \leq c < 1$ we have

$$w(\varepsilon) = |f_n(x) - f(x)| = |x^n - 0| = x^n < \varepsilon, \text{ if } \left(\frac{1}{x}\right)^n > 1/\varepsilon$$

$$n \log(1/x) > \log(1/\varepsilon), \text{ if } n > \frac{\log(1/\varepsilon)}{\log(1/x)}$$

Notice that the maximum value of $\frac{\log(1/\varepsilon)}{\log(1/x)}$ is

$$\frac{\log(1/\varepsilon)}{\log(1/x)} = \varepsilon = m(\varepsilon).$$

$\Rightarrow$ for the given $\varepsilon > 0$ and for all $x \in (0,1)$. $\exists$ $m(\varepsilon) \ni$
$|f_n(x) - f(x)| < \varepsilon, \ \forall \ n > \varepsilon$

Clearly, for $x = 0$.
$|f_n(x) - f(x)| = 0 < \varepsilon, \ \forall \ n \geq 1$

Hence $\{f_n\}$ converges uniformly on $[0, c]$, $c < 1$.

However, we see that for $x = 1$,
$\frac{\log(1/\varepsilon)}{\log(1/x)} \longrightarrow \infty$. this means that $\{f_n\}$ has not converge uniformly on $[0,1]$.

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Test for Uniform Convergence of sequence of functions

THEOREM: (M_n-test) Let $\{f_n\}$ be a sequence of functions such that $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in [a,b]$, and let
$M_n = \sup_{a \le x \le b} |f_n(x) - f(x)|$. then $\{f_n\}$ converges uniformly to f on $[a,b]$ if and only if $\lim_{n\to\infty} M_n = 0$

proof: Necessary: suppose that $f_n \xrightarrow[n\to\infty]{} f$ on $[a,b]$ this means for every $\varepsilon > 0$ and for all $x \in [a,b]$ we can find a positive integer $\tau(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon, \ \forall \ n \geq \tau(\varepsilon) \quad \text{——— (1)}$$

Taking supremum over all $x \in [a,b]$ in (1) gives

$$M_n = \sup_{a \le x \le b} |f_n(x) - f(x)| < \sup(\varepsilon) = \varepsilon$$

i.e $M_n < \varepsilon \ \forall \ n \geq \tau(\varepsilon)$

$$\Rightarrow \lim_{n\to\infty} M_n = 0 \quad \text{——— (2)}.$$

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Sufficiency: suppose that equation (*) holds. then for any $\varepsilon > 0$ $\exists$ a natural number $\tau(\varepsilon) \ni |M_n - 0| < \varepsilon \ \forall \ n \geq \tau(\varepsilon)$

i.e. $M_n < \varepsilon \ \forall \ n \geq \tau(\varepsilon)$

$\Rightarrow \sup |f_n(x) - f(x)| < \varepsilon, \ \forall \ n \geq \tau(\varepsilon), \ x \in [a,b]$

$\Rightarrow |f_n(x) - f(x)| < \varepsilon, \ \forall \ x \in [a,b], \ \forall \ n \geq \tau(\varepsilon)$

$\Rightarrow \{f_n\}$ converges to $f(x)$ on $[a,b]$ uniformly □

Example 1: show that the sequence $\{f_n\}$ whose $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is not uniformly convergent on any interval containing zero.

Solution

The point wise limit is given by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$$

Next, we find the maximum value of $f_n(x)$ for that, we see that

$$f_n'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$

at turning point, $f_n'(x) = 0$

$\Rightarrow n(1-n^2x^2) = 0$

$\Rightarrow x = 1/n$.

$$\text{Max}\, f_n(x)\Big|_{x=1/n} = \frac{n(1/n)}{1+n^2(1/n)^2} = \frac{n(1/n)}{1+n^2(1/n^2)}$$

$$= 1/2$$

$$\therefore M_n = \sup \left| f_n(x) - f(x) \right|$$

$$M_n = \sup_{x\in[a,b]} \left| \frac{nx}{1+n^2x^2} - 0 \right|$$

$$M_n = \sup_{x\in[a,b]} \left| \frac{nx}{1+n^2x^2} \right| = \frac{1}{2} \not\to 0 \quad \text{as } n\to\infty$$

Hence, $\{f_n\}$ does not converges uniformly On any interval $[a,b]$ Containing zero.

i.e $M_n = 1/2$

$$\therefore \lim_{n\to\infty} M_n = 1/2 \neq 0.$$

Ex 1: i) Show that the sequence $\{f_n\}$ Where $f_n(x) = \frac{x}{n+x}$ is Uniformly Convergent on $[0, k]$, $k<\infty$ but only pointwise Convergent When the Interval extends to ∞.

ii) show that the sequence $\left\{\frac{nx}{1+n^3x^2}\right\}$ Converges uniformly to zero for $0\leq x\leq 1$.

iii) show that the sequence $\{nx(1-x^2)\}$ and $\{nx^2(1-x^2)\}$ are not uniformly Convergent on $[0,1]$.

Example prove that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx^2}$, $x \in \mathbb{R}$ Converges uniformly on any closed Interval D of $\mathbb{R}$:

soln The pointwise limit $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$

It is easy to see that $f(x)$ attains its maximum value $\frac{1}{2\sqrt{n}}$ at $x = \frac{1}{\sqrt{n}}$,

therefore,

$$M_n = \sup_{x\in D} |f_n(x) - f(x)|$$

$$= \sup_{x\in D} \left| \frac{x}{1+nx^2} \right| = \frac{1}{2\sqrt{n}}$$

$$\therefore \lim_{n\to\infty} M_n = 0$$

$$\Longrightarrow f_n(x) \xrightarrow[n\to\infty]{} f(x) \text{ on } D, \text{ by } M_n\text{-test}.$$

Ex 2: 1) show that the following sequence are not uniform Convergent on the Indicated Intervals

i- $\{x^n\}$ on $[0,1]$ ii) $\{e^{-nx}\}$ on $[0,k]$, $k>0$

2) Test the following sequence for uniform Convergence

i) $\left\{\frac{\sin(nx)}{\sqrt{n}}\right\}$, $0 \le x \le 2\pi$ ii) $\left\{\frac{x}{n+x}\right\}$, $0 \le x \le k$, $k>0$

iii) $\left\{\frac{n}{n+k}\right\}$, $0 \le x \le \infty$ iv) $\left\{\frac{n^2 x}{1+n^3x^2}\right\}$, $0 \le x \le 1$

Tests for Uniform Convergence of series function

THEOREM: (Weierstrass M-test): A series of function Σf_n will converge uniformly on $[a,b]$ if there exists a convergent series of positive number ΣM_n, if $\forall x \in [a,b]$ $|f_n(x)| \leq M_n, \forall n$.

proof: Let $\varepsilon > 0$ be given. Since ΣM_n converges

$\Rightarrow \lim_{n\to\infty} M_n = 0$. Hence, for the given $\varepsilon > 0$, we can find a natural number $r(\varepsilon)$ such that

$|M_{n+1} + M_{n+2} + M_{n+3} \dots + M_{n+p}| < \varepsilon, \ \forall n \geq r(\varepsilon), \ p \geq 1$.

Now, for the given $\varepsilon > 0$ and for all $x \in [a,b]$,

$|f_{n+1}(x) + f_{n+2}(x) + \dots f_{n+p}(x)| \leq |f_{n+1}(x)| + |f_{n+2}(x)| + |f_{n+3}(x)| + \dots |f_{n+p}(x)|$

$\leq M_{n+1} + M_{n+2} + \dots M_{n+p}$

$< \varepsilon, \ \forall n \geq r(\varepsilon)$

$\Rightarrow \Sigma f_n$ converges uniformly on $[a,b]$.

REMARK: 1) The converse of the above theorem is not always true, i.e. non-convergence of ΣM_n does not imply non-convergence of Σf_n.

2) Series that satisfy Weierstrass M_n-test are sometimes called <u>normally convergent</u> series to emphasize the fact that such series are both uniformly and absolutely convergent.

Examples 1: Test for uniform Convergence, the series

i. $\sum r^n \cos(n\theta)$, ~~and~~ ii) $\sum r^n \sin(n\theta)$

iii. $\sum r^n \sin(n\theta)$ $0 < r < 1$

soln i) Let $\sum f_n = \sum r^n \cos(n\theta)$

Now $|f_n(x)| = |r^n \cos(n\theta)| \leq r^n = M_n$.

i.e $\sum M_n = \sum r^n$

We see that $\lim_{n\to\infty} r^n = 0$ because $0 < r < 1$

$r \in (0,1) \Rightarrow r = \frac{1}{k>1}, \Rightarrow r^n = \left(\frac{1}{k}\right)^n$

$\Rightarrow \lim_{n\to\infty} r^n = \lim_{n\to\infty} \frac{1}{k^n} = \lim_{n\to\infty} = \frac{1}{\infty} = 0$.

It is easy to see that by Cauchy root test, $\sum r^n$ is Convergent. Hence $\sum r^n \cos(n\theta)$ Converges Uniformly by Weierstrass M-test.

for (ii) and (iii) the whole idea is the same.

Example 2: Test for uniform Convergence, the series.

i) $\sum \frac{\sin(x^2+n^2x)}{n(n+1)}$, $x \in \mathbb{R}$

ii) $\sum \frac{(n\theta)}{n^p}$, $p>1$, iii) $\sum \frac{(-1)^n x^{2n}}{n^p(1+x^{2n})}$, $p>1$

~~soln~~ (i)

Let $\sum f_n(x) = \sum \frac{\sin(x^2+n^2x)}{n(n+1)}$

Now $|f_n(x)| = \left|\frac{\sin(x^2+n^2x)}{n(n+1)}\right|$

$\leq \frac{1}{n(n+1)} = \frac{1}{n^2+n} < \frac{1}{n^2} = M_n$

i.e $\sum M_n = \sum \frac{1}{n^2}$

Clearly, $M_n \xrightarrow{n\to\infty} 0$ and $\sum \frac{1}{n^2}$ Converges by Ratio test. Therefore, $\sum f_n$ Converges Uniformly $\forall x \in \mathbb{R}$.

for (ii) and (iii) the whole color is the same.

Example 3: Show that the series $\sum \frac{x}{n^p + x^2 n^2}$ Converges Uniformly over any finite Interval $[a, b]$ for $p > 1$, $q \geq 0$

<u>Soln</u> let $\sum f_n(x) = \sum \frac{x}{n^p + x^2 n^2}$

$$\text{now, } |f_n(x)| = \left| \sum \frac{x}{n^p + x^2 n^2} \right|$$

$$\leq \left| \frac{x}{n^p} \right| = \frac{|x|}{|n^p|} = \frac{|x|}{n^p} \leq \frac{b}{n^p}$$

Obviously, $\lim_{n\to\infty} \frac{b}{n^p} = 0$, and the series $b \sum \frac{1}{n^p}$ is Convergent, being a p-series with $p > 1$.

therefore $\sum \frac{x}{n^p + x^2 n^2}$ is uniformly Convergent on $[a, b]$ by weierstrass M-test.

<u>Quiz Test Question</u>

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1) Discuss the uniform Convergence of the Series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^5}{1+x^8} + \cdots, \quad x \in [-1/2, 1/2]$$

Hint $f_n(x) = \frac{2^n x^{2n-1}}{1+x^{2n}}$

By using Weistrass M_n-test

2) Show that the sequence of function $\{f_n\}$ where $f_n(x) = \frac{n^2 x}{1+n^2 x^2}$ is not uniformly Convergent on $[0, 1]$ using M_n-test.

DEF : A sequence $\{x_n\}$ of numbers is said to be monotonic Increasing if $x_{n+1} \geq x_n, \forall n$. And (x_n) is monotonic decreasing if $x_{n+1} \leq x_n$ $\forall n$. (x_n) is said to be monotone if it is either monotonic Increasing or decreasing.

DEF : A sequence (f_n) of functions is said to be uniformly bounded on $[a,b]$ if there exist a number $\tau \in N$ such that for all $x \in [a,b]$ and $n \in N$, $|f_n(x)| \leq \tau$.

DEF : A function $f_n(x)$ is said to be positive if $f_n(x) \geq 0, \forall x$.

LEMMA : (Abel's Lemma): If $\{B_n\}$ is a positive, monotonic decreasing sequence and denoted respectively the least and the greatest values of the sum $\sum_{r=m} U_r$, where $p = m, m+1, m+2 \ldots$ then $B_m \leq \sum_{r=m} U_r \leq B_m g$.

THEOREM : (Abel's theorem) If $B_n(x)$ is a positive monotonic decreasing function of n and $x \in [a,b]$, and if $B_n(x)$ is uniformly bounded on $[a,b]$ and if the series $\sum U_n(x)$ converges uniformly on $[a,b]$, then the series $\sum b_n(x) U_n(x)$ converges uniformly on $[a,b]$

PROOF : Since the function $B_n(x)$ is bounded for all n and $x \in [a,b]$, then there exists a number $g > 0$ such that

$$0 \leq B_n(x) \leq g, \forall n$$

and $\forall x \in [a,b]$.

Again, since $\sum U_n(x)$ is uniformly convergent on $[a,b]$, then its nth partial sum $\sum_{r=n+1}^{n+p} U_n(x)$ is convergent. This means given any $\varepsilon > 0$, there is a natural number α such that

$$\left|\sum_{r=n+1}^{n+p} U_r(x)\right| < \frac{\varepsilon}{g}, \quad \forall\, n \geq \alpha,\ p \geq 1 \qquad (1)$$

Now by Abel's theorem

$$\left|\sum_{r=n+1}^{n+p} b_r(x)\,U_r(x)\right| \leq b_{n+1}(x)\,g$$

$$= b_{n+1}(x) \max_{q=1,\dots,p} \left|\sum_{r=n+1}^{n+q} U_r(x)\right|$$

$$\leq g \cdot \frac{\varepsilon}{g} = \varepsilon$$

$\Rightarrow \sum b_n(x)\,U_n(x)$ converges uniformly on $[a,b]$ □

Example 1: Determine whether the series

$\sum \frac{(-1)^n}{n} |x|^n$ is uniformly convergent on $[-1,1]$ by using Abel's theorem.

Soln: since $b_n(x) = |x|^n$ is positive monotonic non decreasing and uniformly bounded on $[-1,1]$ and the series $\sum U_n = \sum \frac{(-1)^n}{n}$ converges uniformly on $[-1,1]$. It follows from Abel's theorem that $\sum \frac{(-1)^n}{n} |x|^n$ converges uniformly on $[-1,1]$

Example 2: Show that $\sum \frac{b_n}{n^x}$ Converges uniformly on $[0,1]$ If $\sum b_n$ is Convergent.

Soln since $b_n(x) = \frac{1}{n^x}$ is positive, monotone decreasing and uniformly bounded for all n and on $[0,1]$ and since $\sum b_n$ Converges it is uniformly Convergent. It follows by ~~therefore~~ by Abel's theorem that $\sum \frac{b_n}{n^x}$ Converges uniformly on $[0,1]$.

Ex If $\sum b_n$ is Convergent, then use Abel's theorem to show that each of the following series is uniformly Convergent on $[0,1]$.

i) $\sum b_n x^n$

ii) $\sum \frac{b_n x^n}{1+x^n}$

iii) $\sum \frac{b_n x^n}{1+x^{2n}}$

iv) $\sum \frac{n x^n (1-x) b^n}{1+x^n}$

v) $\sum \frac{2n\, b_n x^n (1-x)}{1+x^{2n}}$

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THEOREM (Dirichlet's test)

If $B_n(x)$ is a monotone function and tends uniformly to zero on $[a,b]$ and if there exists a number $K>0$

$$\left|\sum_{r=1}^{n} U_r(x)\right| \leq K, \ \forall n,$$

then the series $\sum b_n(x) U_n(x)$ Converges uniformly on $[a,b]$

proof:
(The whole ideas can be followed from Abel's theorem.)

Example 1

prove that the series $\sum (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval. But it does not converges absolutely, using Dirichlet's test.

solution

let $U_n = (-1)^n$, $n \in \mathbb{N}$. Clearly, $\exists$ $k = 1$ $\ni$.

$$\left| \sum_{r=1}^{n} U_r(x) \right| = 1$$

Take $b_n(x) = \frac{x^2+n}{n^2}$ and let D be the bounded interval. This implies that $\exists$ $K>0$ $\ni$ $|x|<K, \forall x \in D$. Then, we can see that

$$b_n(x) = \frac{x^2+n}{n^2} < \frac{K^2+n}{n^2}$$ is a monotonic decreasing function and tends uniformly to zero (0). Hence, by Dirichlet's test, the series

$\sum b_n(x) U_n(x)$ converges uniformly on D.

for absolute convergence, we see that

$$\left| \sum (-1)^n \frac{x^2+n}{n^2} \right| = \sum \frac{x^2+n}{n^2} \sim \sum \frac{1}{n}$$

$\sum \frac{x^2+n}{n^2} \sim \sum \frac{1}{n}$, which diverges. Hence $\sum (-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely on any bdd interval.

Some properties of Uniformly Convergent series of function

We shall learn that the sufficient condition for a limit function or series to enjoy / inherit all the fundamental properties of a sequence of function or series is that

THEOREM

If a sequence $\{f_n\}$ converges uniformly on $[a,b]$ and x_0 is a point in $[a,b]$ such that

$$\lim_{x \to x_0} f_n(x) = y_n, \quad n \in \mathbb{N}.$$

i- $\{y_n\}$ converges

ii- $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} y_n$

proof;

i) Let $\{f_n\}$ converges uniformly on $[a,b]$. This means $\{f_n\}$ is a Cauchy sequence. Hence, given any $\varepsilon > 0$, we can find a natural number $q(\epsilon) \ni \quad |f_n(x) - f_m(x)| < \varepsilon/2, \ \forall\, n,m \geq q(\epsilon)$ — (i)

Since $\lim_{x \to x_0} f_n(x) = y_n, \ \forall n$. then letting $x \to x_0$ in (i), we get

$$|y_n - y_m| < \varepsilon/2, \ \forall\, n,m \geq q(\epsilon)$$

$\Rightarrow \{y_n\}$ is a Cauchy sequence and hence convergent.

(ii) ~~A~~ssume that $\{f_n\}$ converges uniformly on $[a,b]$. This means for any $\varepsilon>0$, we can find an integer $\beta_1(\varepsilon)\in\mathbb{N}$ such that

$$\left|f_n(x)-f(x)\right| < \varepsilon/3 \, , \forall \, n\geq \beta_1(\varepsilon) \quad \text{———} \quad (a)$$

Since $\{y_n\}$ converges to p, then for a given $\varepsilon>0$, there exists a number $\beta_2(\varepsilon)>0$ such that

$$\left|y_n-p\right| < \varepsilon/3 \, , \forall \, n\geq \beta_2(\varepsilon) \quad \text{———} \quad (b)$$

Let $\alpha = \max\left(\beta_1(\varepsilon), \beta_2(\varepsilon)\right)$

By hypothesis $\lim\limits_{x\to x_0} y_n(x) = y_n \, , \forall \, n$

Therefore $\lim\limits_{x\to x_0} f_\alpha(x) = y_\alpha$. Hence for any $\varepsilon>0$, $\exists \; \delta>0$ such that

$$|x-x_0|<\delta \implies \left|f_\alpha(x)-y_\alpha\right| < \varepsilon/3 \quad \text{———} \quad (c)$$

Therefore, for the given $\varepsilon>0$, and $|x-x_0|<\delta$, we see that

$$\begin{aligned}|f(x)-p| &= \left|f(x)-f_\alpha(x)+f_\alpha(x)-y_\alpha+y_\alpha-p\right| \\ &\leq \left|f(x)-f_\alpha(x)\right|+\left|f_\alpha(x)-y_\alpha\right|+\left|y_\alpha-p\right| \\ &< \varepsilon/3+\varepsilon/3+\varepsilon/3 \\ &= \varepsilon\end{aligned}$$

14/2/22

THEOREM

If a series $\sum_{n=1}^{\infty} f_n$ Converges uniformly to a sum function f on $[a,b]$ and x_0 is a function $[a,b]$ such that $\lim_{n\to\infty} f_n = y_n$, $n \in \mathbb{N}$, then

i. $\sum_{n=1}^{\infty} y_n$ Converges

ii. $\lim_{x\to x_0} f(x) = \sum_{n=1}^{\infty} y_n$

proof (same idea with the previous result).

REMARK

The Consequences of the above theorem is that

$$\lim_{x\to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to x_0} f_n(x)$$

UNIFORM CONVERGENCE & CONTINUITY

Recall:

A function f is said to be continuous at $x_0 \in D(f)$ if given any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that

$|x - x_0| < \delta_\varepsilon \Rightarrow |f(x) - f(x_0)| < \varepsilon, \ \forall x \in D(f)$

THEOREM

If $\{f_n\}$ is a sequence of continuous functions on an interval $[a,b]$ and if $f_n \to f$ uniformly on $[a,b]$, then the limit function f is continuous on $[a,b]$.

proof (exercise).

THEOREM

If a series Σf_n converges uniformly to a sum function f on an interval $[a,b]$ and if f_n is continuous at $x_0 \in [a,b]$ for each n, then the sum function f is also continuous at x_0.

Proof:

Since Σf_n converges uniformly to f on $[a,b]$ then given any $\varepsilon > 0$, we can produce an $\eta \in N \ni$ for all $x \in [a,b]$,

$$\left| \sum_{i=1}^{n} f_i(x) - f(x) \right| < \varepsilon/3, \quad \forall \; n \geq \eta \quad \text{———} \quad (1)$$

In particular, for $x = x_0$ and $n = \eta$, we get

$$\left| \sum_{i=1}^{\eta} f_i(x_0) - f(x_0) \right| < \varepsilon/3 \quad \text{———} \quad (2)$$

Again, since Σf_n is continuous at x_0, the sum function $\sum_{i=1}^{\eta} f_i$ is continuous, because finite sum of continuous functions is always continuous. Therefore, given any $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that

$|x - x_0| < \delta_\varepsilon$ gives

$$\left| \sum_{i=1}^{\eta} f_i(x) - \sum_{i=1}^{\eta} f_i(x_0) \right| < \varepsilon/3 \quad \text{———} \quad (3)$$

Now, for the given $\varepsilon > 0$, with $\delta_\varepsilon > 0$, we see that $|x - x_0| < \delta_\varepsilon$ yields

$$|f(x)-f(x_0)| = \left|f(x)-\sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{n} f_i(x_0) + \sum_{i=1}^{n} f(x_0) - f(x_0)\right|$$

$$\leq \left|f(x)-\sum_{i=1}^{n} f_i(x)\right| + \left|\sum_{i=1}^{n} f(x) - \sum_{i=1}^{n} f_i(x_0)\right| + \left|\sum_{i=1}^{n} f(x_0) - f(x_0)\right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

i.e. $|x-x_0| < \delta$

$\Rightarrow |f(x)-f(x_0)| < \varepsilon \ \forall, x \in [a,b]$ this proves that f is continuous at $x_0 \in [a,b]$. $\square$

REMARK

The Convergence of the above theorem is not always true. that is, there exists series or sequence of continuous terms which have a continuous sum or limit. But which are not uniformly continuous.

However, if the sum or limit function is not continuous then the convergence cannot be uniform on the given interval.

Examples

Show that the following do not converges uniformly on the indicated interval.

i. $\left\{\frac{nx}{1+n^2x^2}\right\}$, on $[0,1]$ $\to$ (pointwise sm

ii. $\sum_{n=1}^{\infty} (1-x)x^n$ on $[0,1]$.

Soln

i) Clearly, $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

Now
$$\left|f_n(x) - f(x)\right| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2} = \frac{1}{nx} < \varepsilon$$

provided $n > \frac{1}{\varepsilon} = m(\varepsilon, x)$

Hence, we conclude that the sequence does not converge uniformly on [0, 1]

ii) Obviously, $f(x) = \begin{cases} 1, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$

We see that $f(x)$ is not continuous. Hence, the series cannot converge uniformly on [0,1].

NOTE

There is a special class of sequence and series of functions for which uniform convergence is equivalent to the continuity of the limit or sum function. In this connection, we have a theorem due to an Italian mathematician called Dini.

THEOREM (Dini's theorem on uniform convergence).
If a sequence of ~~continuous~~ functions $\{f_n\}$ defined on $[a,b]$ is monotonic increasing and converges pointwise to a ~~continuous~~ function f, then the convergence is uniform.

THEOREM (Dini's theorem on Uniform Convergence of Series)

If the sum function f of a series Σf_n with non-negative continuous terms defined on an interval $[a, b]$ is continuous, then the series is uniformly convergent.

REMARK:

If the pointwise limit or sum function is not continuous, then the convergent cannot be uniform.

Example 1

Show that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots$

is not uniformly convergent on $[0, 1]$.

$$\sum^{\infty} f_n = \sum_{n=0}^{\infty} \frac{x^4}{(1+x^4)^n}$$

$a = x^4 \qquad r = \frac{1}{1+x^4} < 1$

for $x \neq 0$,

$$\therefore f(x) = S_\infty = \frac{a}{1-r} = \frac{x^4}{1 - \frac{1}{1+x^4}} = 1 + x^4$$

for $x = 0$, $\Sigma f_n = 0 = f(x)$

for $x \neq 0$, $\Sigma f_n = 1 + x^4 = f(x)$

$$\therefore f(x) = \begin{cases} 1 + x^4, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

We see that the sum function is not continuous on [0, 1]. Hence the series cannot converge uniformly on [0, 1].

Example 2

Show that $\sum \frac{x}{(nx+1)\{(n-1)x+1\}}$ is called telescoping series is uniformly convergent on any interval [a, b], $0 < a < b$ but only pointwise on [0, b].

Soln

Let $$f_n(x) = \frac{x}{(nx+1)\{(n-1)x+1\}}$$

$$= \frac{1}{(n-1)x+1} - \frac{1}{(nx+1)} \quad \text{(by partial fraction)}$$

The nth partial sum of $\sum f_n$ is given by

$$S_K(x) = \sum_{n=1}^{K} f_n(x) = \sum_{n=1}^{K}\left[\frac{1}{(n-1)x+1} - \frac{1}{(nx+1)}\right]$$

$$= \left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) + \left(\frac{1}{2x+1} - \frac{1}{3x+1}\right) + \dots + \left(\frac{1}{(K-1)x+1} - \frac{1}{Kx+1}\right)$$

$$= 1 - \frac{1}{Kx+1}$$

$$\therefore \lim_{K\to\infty} S_K(x) = \lim_{K\to\infty}\left[1 - \frac{1}{Kx+1}\right] = 1 = f(x)$$

$$\therefore f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$\therefore$ The sum function is discontinuous on [0, b] and therefore, convergence cannot be uniform on [0, b]. i.e. $\sum f_n$ is only pointwise on [0, b]

Now, for $x \neq 0$, i.e $x \in [a, b]$ with $0 < a < b$. Let $\varepsilon > 0$ be given. Then

$\frac{1}{nx} < \varepsilon$

$\varepsilon nx = 1$

$n > \frac{1}{\varepsilon x}$

$\varepsilon(nx+1) = 1$

$nx\varepsilon + \varepsilon = 1$

$nx\varepsilon = 1 - \varepsilon$

$n = \frac{1-\varepsilon}{x\varepsilon}$

$$\left| S_n(x) - f(x) \right| = \left| \left(1 - \frac{1}{nx+1}\right) - 1 \right| = \frac{1}{nx+1} < \frac{1}{nx} < \varepsilon$$

provided $n > \frac{1}{\varepsilon x}$.

Observe that $\frac{1}{\varepsilon x}$ decreases with x. Let its minimum be $\frac{1}{a\varepsilon} = \alpha$ in $[a, b]$. Therefore, for all $x \in [a, b]$, there exists an $\alpha \in N$ such that $|S_n(x) - f(x)| < \varepsilon$, $\forall\ n \geq \alpha$.

Hence, the series converges uniformly on $[a, b]$, $0 < a < b$.

9th-Jan-2023

After 11 month at home

Uniform Convergence and Integrability

Def: Let $[a, b]$ be a given closed interval. A partition of $[a, b]$ is a finite set P of numbers $x_0, x_1, \ldots x_n$, where $a = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n = b$.

The intervals $[x_0, x_1], [x_1, x_2] \ldots [x_{n-1}, x_n]$ are the sub intervals of $[a, b]$. Note that the length of the interval $[a, b]$ is given by

$$l[a, b] = b - a.$$

We denote the lengths of the sub intervals of $[a, b]$ by Δx_i $(i = 1, 2, 3 \ldots, n)$. That is

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, 3 \ldots, n).$$

Let f be a bounded function on $[a,b]$. and M_i, m_i denote respectively the supremum and the Infimum of f on $[a,b]$. Then, Consider the sums

$$U(P,f) = \sum_{i=1}^{m} M_i \Delta x_i \quad \text{------} (i)$$

$$L(P,f) = \sum_{i=1}^{m} m_i \Delta x_i \quad \text{------} (ii)$$

The sums in (i) and (ii) are respectively called the upper and the lower Darboux sums of f with respect to the partition P on $[a,b]$.

Def: For any partition P of $[a,b]$, the length of the maximum sub-interval is called the **mesh** or the **norm** of the partition P, and is denoted by $\mathcal{U}(P)$. That is, $\mathcal{U}(P) = \max_{1 \leq i \leq n} \Delta x_i$

$$= \max \{ x_i - x_{i-1} ; 1 \leq i \leq n \}.$$

THEOREM: Necessary and sufficient Conditions for Integrability

A bounded function f on $[a,b]$ is Integrable on $[a,b]$ Iff for every $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

By ( ) $\lim \left(U(P,f) - L(P,f) \right) = 0$

p.t.o

THEOREM: If a sequence $\{f_n\}$ of functions converges uniformly on $[a,b]$ to a limit function f, and each f_n is integrable on $[a,b]$, then the limit function is integrable on $[a,b]$ and the sequence $\{\int_a^x f_n\,dt\}$ converges uniformly to $\{\int_a^x f_n\,dt\}$.

i.e $\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt, \ \forall x \in [a,b]$.

proof: Suppose that $\{f_n\}$ converges to f on $[a,b]$. Then by definition for every $\epsilon > 0$, and $\forall x \in [a,b]$ we can find a positive integer $\eta \ni$

$$|f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}, \ \forall n \geq \eta \quad \text{---} (1)$$

In particular for $n = \eta$, we get

$$|f_\eta(x) - f(x)| < \frac{\epsilon}{3(b-a)} \quad \text{------} (2)$$

for this fixed η, and since f_η is integrable we can choose a partition $P\eta$ of $[a,b] \ni$

$$U(P, f_\eta) - L(P, f_\eta) < \epsilon/3 \quad \text{---} (3)$$

from (2), we can write:

$$-\frac{\epsilon}{3(b-a)} < f_\eta(x) - f(x) < \frac{\epsilon}{3(b-a)} \quad \text{---} (4)$$

Now from LHS of (4) we have

$$-\frac{\epsilon}{3(b-a)} < f_\eta(x) - f(x) \Rightarrow f(x) - \frac{\epsilon}{3(b-a)} < f_\eta(x).$$

$\Rightarrow f(x) < f_\eta(x) + \frac{\epsilon}{3(b-a)}$ --- (5)

$\Rightarrow U(P,f) < U(P,f_\eta) + \epsilon/3$ --- (6)

Again, from RHS of (4) we get:

$f_\eta(x) - f(x) < \frac{\epsilon}{3(b-a)} \Rightarrow f_\eta(x) - \frac{\epsilon}{3(b-a)} < f(x)$

$\Rightarrow f(x) < f_\eta(x) - \frac{\epsilon}{3(b-a)}$

$\Rightarrow L(P,f) > L(P,f_\eta) - \epsilon/3$ ------ (7)

from (6) and (7), we have

$U(P,f) - L(P,f) < U(P,f_\eta) - L(P,f_\eta) + 2\epsilon/3$

$\Rightarrow U(P,f) - L(P,f) < \epsilon/3 + 2\epsilon/3 = \epsilon$

$\therefore f$ is Integrable on $[a,b]$.

Now, to show that $\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt$,

let $\{f_n\}$ Converges Uniformly to f on $[a,b]$.

$\Rightarrow$ for every $\epsilon > 0$, we can find a positive Integer $\eta \ni \forall x \in [a,b]$,

$\therefore |f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}, \forall n \geq \eta$ --- (8)

Then for all $x \in [a,b]$ and for $n \geq \eta$, we get

$$\left|\int_a^x f\,dt - \int_a^x f_n\,dt\right| = \left|\int_a^x (f\,dt - f_n\,dt)\right|$$

$$\leq \int_a^x |f - f_n|\,dt = |f - f_n|(x-a)$$

$$\leq |f - f_n| \sup_{x\in[a,b]} (x-a)$$

$$\leq |f - f_n|(b-a)$$

$$\leq \frac{\epsilon}{3(b-a)}(b-a) = \epsilon/3 < \epsilon.$$

i.e. $\left|\int_a^x f\,dt - \int_a^x f_n\,dt\right| < \epsilon, \forall x \in [a,b]$

$\Rightarrow \lim \int_a^x f_n\,dt = \int_a^x f\,dt$ $\square$.

REMARK: The Converse of the above theorem need not to be true.
That is the limit of the function f is Integrable; the sequence of function (f_n) is not necessary uniformly convergent. Note however that if the limit function f is not Integrable, then the convergence of $\{f_n\}$ cannot be uniform.

12/1/2023

THEOREM: If a series $\sum_{n=1}^{\infty} f_n$ Converges to a limit function f on $[a,b]$ and each f_n is Integrable for each n, then f is also Integrable on $[a,b]$ and $\sum_{n=1}^{\infty}\left(\int_{\infty}^{x} f_n\,dt\right)$ Converges uniformly to $\int_{\infty}^{x} f\,dt$
that is, $\int_{\infty}^{x} f\,dt = \sum_{n=1}^{\infty}\left(\int_{\infty}^{x} f_n\,dt\right)$

$$= \int_{\infty}^{x} f_1\,dt + \int_{\infty}^{x} f_2\,dt + \cdots$$

In this case, we say that the series $\sum_{n=1}^{\infty} f_n\,dt$ is term by term Integrable.

proof: (The whole idea is the same when the series is replaced with $\{f_n\}_{n\in N}$)

REMARK: The Converse of the theorem is not always true.
that is, $\sum f_n$ may Converge to an Integrable limit f, but the Convergence $\sum f_n$ may not be uniform.

But, if the pointwise limit is not integrable or integrable but the integral is not equal to the sum of the series, then term-by-term integration is not possible and hence the convergence of the series is not uniform.

Example 1

Show that for the series $1-x+x^2-x^3+x^4+\dots=\frac{1}{1+x}$, $0\le x\le 1$, term-by-term integration is possible but the series does not converge uniformly.

Solution

Integrating the R.H.S of (1) over $[0,1]$, to have $\int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1 = \ln(2)$

~~Again~~, Integrating the LHS of (1) over $[0,1]$,

$\int_0^1 (1-x+x^2-x^3+\dots)\,dx = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$

We know that $\ln(2) = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$

$\therefore$ LHS = RHS $\Rightarrow$ term-by-term integration is possible. However, it is clear that the series does not converge uniformly on $[0,1]$.

Example 2

Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$, $n=1,2,3\dots$ cannot converge uniformly on $[0,1]$.

P.T.O

Soln: It is enough to show that $\cancel{\int_0^1 f(x)\,dx} \neq$
$\int_0^1 f(x)\,dx \neq \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$.

Now,

$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty}\frac{nx}{e^{nx^2}} = 0$

i.e. $f(x) = 0$

We ~~seg~~ say,

$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0.$

Similarly, $\int_0^1 f_n(x)\,dx = \int_0^1 nxe^{-nx^2}\,dx = \frac{1}{2}(1-e^{-n})$

$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{1}{2}(1-e^{-n}) = \frac{1}{2}$.

$\therefore \int_0^1 f(x)\,dx = 0 \neq \frac{1}{2} = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$.

Hence, the sequence does not converge uniformly

16/1/2023

UNIFORM CONVERGENCE AND DIFFERENTIABILITY.

Recall: (Lagrange mean value theorem / first mean value theorem). If a function f defined on $[a, b]$ is:

(L_1) Continuous $[a, b]$

(L_2) differentiable on (a, b),

Then there exists a real number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Recall: A function is said to be differentiable at c if $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

THEOREM: Let $\{f_n\}$ be a sequence of differentiable fxns on $[a, b]$ such that it converges at least at one point x_0 in $[a, b]$. If the sequence of differentials $\{f'_n\}$ converges uniformly to a limit fxn p on $[a, b]$, then the given sequence $\{f_n\}$ converges uniformly to a fxn f on $[a, b]$ and

$$f'(x) = p(x), \forall x \in [a, b].$$

proof: Let $\epsilon > 0$ be given. By the convergence of $\{f_n(x_0)\}$ and $\{f'_n\}$, then for the given $\epsilon > 0$, we can find a natural number $N \ni$ for all $x \in [a, b]$, we have

$$|f_n(x_0) - f_m(x)| < \epsilon/2 \quad \forall n, m \geq N \text{ --- (1) and}$$

$$|f'_n(x) - f_m(x)| < \epsilon/2(b-a), \quad \forall n, m \geq N \text{ --- (2)}$$

Since $(f_n - f_m)$ is differentiable and hence continuous on $[a, b]$ then by the Lagranges MVT, for any two point $x, t \in [a, b]$ there exist a real number $c \in (x, t)$ such that

$$\frac{|f_n(x) - f_m(x) - (f_n(t) - f_m(t))|}{|x - t|} = |f'_n(c) - f'_m(c)|$$

$$\Rightarrow |f_n(x) - f_m(x) - f_n(t) + f_m(t)| = |x-t||f_n'(c) - f_m'(c)|$$

$$\leq \sup_{x,t \in [a,b]} |x-t||f_n'(c) - f_m'(c)|$$

$$= |b-a||f_n'(c) - f_m'(c)|$$

$$< |b-a| \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{2} \quad \text{---} (3)$$

Hence, using (2) and (3), we get

$$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f(x_0) - f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$\Rightarrow$ The sequence $\{f_n\}$ converges uniformly on $[a,b]$ to a function f (say).

Now for $x, t \in [a,b]$, consider the auxiliary functions defined as follows:

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad x \neq t \text{ --- } (4)$$

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \quad x \neq t \text{ --- } (5)$$

Since f_n is differentiable for each n, then from (4), we see that

$$\lim_{t \to x} \phi(t) = \lim_{t \to x} \frac{f_n(t) - f_n(x)}{t - x} = f_n'(x) \quad \text{---} (6)$$

$\therefore \left| \phi_n(t) - \phi_m(t) \right| = \left| f_n(t) - f_m(x) - f_m(t) + f(x) \right|$

$$< \frac{\epsilon}{2(b-a)} < \epsilon/2 < \epsilon \quad \forall\, n,m < \epsilon$$

$\Rightarrow \{\phi_n'(t)\}$ Converges Uniformly on $[a,b]$ since $\{f_n\}$ also Converges Uniformly on $[a,b]$ to f, it follows from (4) that

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} f_n(t) - f_n(x) = \frac{f(t) - f(x)}{t-x} = \phi(t)$$

THEOREM Therefore, $\{\phi_n(t)\}$ Converges Uniformly to $\phi(t)$.

Now recall that if a sequence $\{f_n\}$ Converges Uniformly to f on $[a,b]$ and $x_0 \in [a,b]$ such that $\lim_{x\to x_0} f_n(x) = y_n$, then $\{y_n\}$ Converges and

$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty} y_n$$

Applying this result to the Uniformly Convergent sequence $\{\phi_n(t)\}$ and Using (6) we get

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x) = \rho(x) \quad \text{---} \quad (7)$$

$$\Rightarrow \lim_{t\to x} \phi(t) \text{ exists.}$$

Hence from (5), we get

$$\lim_{t\to x} \phi(t) = \lim_{t\to x} \frac{f(t) - f(x_0)}{t-x} = f'(x) \quad \text{---} \quad (8)$$

That is f is differentiable.

Thus by uniqueness of limit it follows from (7) and (8) that $f'(x) = p(x)$. $\square$

THEOREM: If a series $\sum f_n$ of differentiable functions converges pointwise to f on $[a,b]$ and each f_n' is continuous and the series $\sum f_n'$ converges uniformly to p on $[a,b]$, then the given series converges uniformly to f on $[a,b]$ and $f'(x) = p(x)$ $\forall \in [a,b]$.

proof: Since the series $\sum f_n'$ of continuous fxns converges uniformly to p on $[a,b]$ therefore the sum function p is also continuous on $[a,b]$. Consequently $\int_a^x p(t)\,dt$ is differentiable and $\frac{d}{dx}\int_a^x p(t)\,dt = p(x)$ $\forall x \in [a,b]$ — (1)

for every $x \in [a,b]$ let $f(x) = \sum_{n=1}^{\infty} f(x)$ —— (2)

Since each function f_n' being continuous is integrable on $[a,b]$, then by the fundamental theorem of calculus

i) $$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a), \quad \forall x \in [a,b] -$$

$$\sum_{n=1}^{\infty} f_n'(t)\,dt = f(x) - f(a), \quad \forall x \in [a,b] --- (3)$$

Again, since the series $\sum f_n'$ of integrable functions converges uniformly in p on $[a,b]$. Therefore term-by-term integration is valid for

$$\int_a^x p(t)\,dt = \sum_{n=1} \int_a^x f_n(t)\,dt, \quad \forall\ x \in [a,b]$$

$$= f(x) - f(a) \quad \text{———} \quad (4)$$

$$\frac{d}{dx}\int_a^x p(t)\,dt = \frac{d}{dx}\left[f(x) - f(a)\right]$$

$\Rightarrow p(x) = f'(x) - 0 = f'(x),\ \forall\ x \in [a,b]$, or equivalently $\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$

i.e term-by-term differentiation is valid $\square$.

Example

Show that the sequence $\{f_n\}$ where

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n} \\ -n^2 x + 2n, & 1/n \le x \le 2/n \\ 0 & 2/n \le x \le 1 \end{cases}$$

is not uniformly converged on $[0,1]$.

Solution

For all $x \in [0,1]$, we have

$$f(x) = \lim_{n \to \infty} f_n(x) = 0$$

Observe that each function f_n and f are continuous on $[0,1]$. Also

$$\int_0^1 f(x) = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (-n^2 x + 2n)\,dx + \int_{2/n}^{1} 0\,dx = 1/2$$

But $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

$$\therefore \lim_{n \to \infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 f(x)\,dx.$$

Hence the sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

Quiz

Show that the sequence $\{f_n\}$ defined on $[0,1]$ by the $f(x) = \begin{cases} n(1-nx), & 0 < x < 1/n \\ 0, & \text{otherwise} \end{cases}$.

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Functions of ~~bd~~ bounded variation

Recall that a function f on $[a,b]$ is said to be bounded if $\exists\ M>0 \ni |f(x)| = M \ \forall x \in [a,b]$

Recall: A partition of a $[a,b]$ is a finite set of points $x_0, x_1, x_2 \ldots x_n \ni$

$$a = x_0 \leq x_1 \leq \ldots \leq x_n = b.$$

Let the function f be defined on $[a,b]$ and $P = \{a = x \leq x_1 \leq x_2 \leq \ldots \leq x_n = b\}$ be any partition of $[a,b]$.

Consider the sum

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|. \quad \text{—————} \quad (i)$$

Since each of such sums in (i) corresponds to the partition P of $[a,b]$, then the set of such sum is infinite. If the set has an upper bound, then the function f is said to be of bounded variation, and the supremum of such set is called the total variation of f over $[a,b]$ denoted by $V(f,a,b)$. i.e

$$V(f,a,b) = \sup_{p} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

Remark: from the above definition, it is clear that a function f on $[a,b]$ is of bounded variation on $[a,b]$ if its total variation on $[a,b]$ is finite. that is

$$V(f,a,b) < \infty .$$

Recall: A function f on $[a,b]$ is called monotone if it is either increasing or decreasing. That is f is monotonic increasing on $[a,b]$ if

$$f(a) \le f(b), \text{ whenever } a \le b .$$

On the other hand, f is monotonic decreasing if $a \le b \Rightarrow f(a) \ge f(b)$.

Example

A bounded monotonic function is of bounded variation.

proof:

Let f be a monotonic increasing fxn on $[a,b]$ and $P = \{a = x_0 \le x_1 \le \cdots \le x_n\}$ be

a partition of $[a,b]$
Note that f being monotonic increasing on $[a,b]$ implies that $f(a) \le f(b)$. Now,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = f(b) - f(a)$$

$$\therefore \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(b) - f(a)]$$

$$= f(b) - f(a) < \infty$$

i.e. $V(f, a, b) < \infty$. Hence f is of bounded variation on $[a,b]$.

<u>Example</u>

If the derivative f' of f exist and is bounded on $[a,b]$, then the function f is of bounded variation.

<u>Solution</u>

Since f' is of bounded, then $\exists\ M > 0$ $\ni |f'(x)| \le N$, $\forall x \in [a,b]$.

Since f' exist on (a,b), then f is continuous on $[a,b]$.

$\therefore$ By mean value theorem (MVT) $\exists\ c \in (a,b)$ $\ni$ $f'(c) = \dfrac{f(b) - f(a)}{b-a}$, $a \ne b$ —— (1)

Let $p = \{a = x_0 \leq x_1 \leq x_2 \ldots \leq x_n = b\}$ be a partition of $[a, b]$. Now, we need to show that the total variation of f on $[a, b]$ is finite. for this, Consider,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |x_i - x_{i-1}| |f'(c)|$$

$$\leq (b-a) M$$

$$\therefore V(f) = \sup_p \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$\leq \sup_p (b-a) M < \infty.$$

Exercise

A function of bounded variation is necessarily bounded.

Soln

suppose the fxn is of bounded variation then $V(f) = \sum |f(x_i) - f(x_{i-1})| < \infty$

And now, we need to show that the function f is bounded, i.e

$\exists\, M > 0 \ni |f(x)| \leq M$

$|f(b) - f(a)| = |f(b) - f(x) + f(x) - f(a)|$

$\leq |f(b) - f(x)| + |f(x) - f(a)|$

$\therefore |f(x)| \Rightarrow \leq V(f, a, b)$

$= M$

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Some properties of function of Bounded Variation

THEOREM: The sum of two functions of bounded variation is of bounded variation

proof:

Let f and g be any two functions of bounded variation on $[a,b]$, and $P = \{a = x_0 \le x_1 \le \dots \le x_n = b\}$ be a partition of $[a,b]$.

This means that $V(f) < \infty$ and $V(g) < \infty$.

we are to show that $V(f+g) < \infty$. so, we consider

$$\sum_{i=1}^{n} \left|(f+g)(x_i) - (f+g)(x_{i-1})\right| = \sum_{i=1}^{n} \left|[f(x_i) - f(x_{i-1})] + [g(x_i) - g(x_{i-1})]\right|$$

$$\le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| - (i)$$

Taking sup of (i) over P, to get

$$\sup_P \sum_{i=1}^{n} \left|(f+g)(x_i) - (f+g)(x_{i-1})\right| \le \sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sup_P \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$\Rightarrow V(f+g) \le V(f) + V(g) < \infty$

$\Rightarrow f+g$ is of bounded variation

THEOREM: The product of two functions of bounded variation is of bounded variation.

proof (exercise).

Variation function

Let f be a function of bounded variation on $[a,b]$ and x a point of $[a,b]$. Then the ~~total~~ variation of f, $V(f,a,x)$ on $[a,x]$, which is clearly a function of x, is called the variation function of x.

We denote the variation function of f on $[a,x]$ by $\check{f}(x)$.

$$\text{i.e } \check{f}(x) = V(f,a,x)$$

If x_1 and x_2 are any two points of $[a,b]$ with $x_2 \geq x_1$, then we observe that

$$0 \leq |f(x_2) - f(x_1)| \leq V(f, x_1, x_2)$$
$$= V(f,a,x_2) - V(f,a,x_1)$$
$$= \check{f}(x_2) - \check{f}(x_1)$$

i.e $\check{f}(x_2) - \check{f}(x_1) \geq 0$ or $\check{f}(x_2) \geq \check{f}(x_1)$

This shows that the variation function $\check{f}(x)$ is a monotonic increasing function.

THEOREM: (Jordan's theorem)

Every function of bounded variation can be expressed as a difference of two monotone increasing functions.

i.e, if f is a function of bounded variation, ~~there~~ there exist two monotone increasing function p and q (say) such that

$$f(x) = p(x) - g(x), \forall x \in [a,b].$$

proof:

Suppose f is of bounded variation on $[a,b]$. Let p and q be any two function defined on $[a,b]$. Then, we are to show that p and q are monotone Increasing and $f(x) = p(x) - q(x)$ $\forall x \in [a,b]$

Let p and q be defined as follows:

$$p(x) = \frac{1}{2}\left[\check{f}(x) + f(x)\right) \text{ and } q(x) = \frac{1}{2}\left(\check{f}(x) - f(x)\right)$$

note that $f(x) = p(x) - q(x)$

Now, let x_1 and x_2 be any two points of $[a,b]$ with $x_2 \geq x_1$. Then,

$$p(x_2) - p(x_1) = \frac{1}{2}\left(\check{f}(x_2) + f(x_2)\right) - \frac{1}{2}\left(\check{f}(x_1) - f(x_1)\right)$$

$$= \frac{1}{2}\left[\check{f}(x_2) - \check{f}(x_1)\right] + \frac{1}{2}\left[f(x_2) - f(x_1)\right]$$

$$= \frac{1}{2}\left[\check{f}(x_2) - \check{f}(x_1) + f(x_2) - f(x_1)\right]$$

$$= \frac{1}{2}\left[V(f, x_1, x_2) + f(x_2) - f(x_1)\right] \text{ --- (i)}$$

we know that $V(f, x_1, x_2) \geq |f(x_2) - f(x_1)|$

therefore (i) becomes

$$p(x_2) - p(x_1) \geq \frac{1}{2}\left[2\,|f(x_2) - f(x_1)|\right]$$

$$= |f(x_2) - f(x_1)| \geq 0$$

i.e $p(x_2) \geq p(x_1)$, proving that p is monotone Increasing

Similarly, we can show that $q(x_2) \geq q(x_1)$ $\square$

Note, for every function of bounded variation f, the two monotone increasing fxn p and q such that $f(x) = p(x) - q(x)$, are respectively called the positive and negative parts of variation function of f.

Example

Determine whether or not, the following fxn f is of bounded variation on [a, b],

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Soln

Recall if the derivative of a function f exist on [a,b] and is bounded, then f is of bnded variation

Now, for $x \neq 0$, we have

$$f'(x) = 2x \sin(1/x) + x^2 \cos(1/x) \frac{d}{dx}(1/x)$$

$$= 2x \sin(1/x) - x^2 \cos(1/x) \cdot 1/x^2$$

$$= 2x \sin(1/x) - \cos(1/x)$$

$\therefore$ $f'(x)$ exists on [0, 1]

Now for $f'(0) = 0$, f' is obviously bnded.

Now, for $x \neq 0$,

$$|f'(x)| = |2x\sin(1/x) - \cos(1/x)|$$
$$\leq |2x\sin(1/x)| + |\cos(1/x)|$$
$$\leq 2(1) + 1 = 3$$

i.e. $|f'(x)| \leq 3 = \alpha$

$\therefore$ f is of bounded variation [0,1].

Exercises

1) Determine whether or not f is of bounded variation on [0,1], where $f(x) = \sqrt{x}\sin(1/x)$, $x \neq 0$, 0, for $x = 0$.

2) $f(x) = \begin{cases} x^2\sin(1/x^2) & x \neq 0 \\ 0, & x = 0. \end{cases}$

✓3) If f is of bounded variation on $[a,b]$, and if there exists a positive number η such that $|f(x)| \geq \eta$, for all $x \in [a,b]$, then show that $1/f$ is of bounded variation on $[a,b]$.

✓4) If f is of bounded variation on $[a,b]$ then show that it is also of bounded variation [a,c] and [c,b].

Riemann Stieltjes Integration

Recall that for any partition P of $[a,b]$, then length of largest sub interval is called the norm or the mesh of P and is denoted by $\mu(P)$ or $\|P\|$ or simply μ. That is

$$\mu(P) = \|P\| = \max_{1 \le i \le n} \Delta x_i, \quad \Delta x_i = x_i - x_{i-1}, \quad i = 1,2,3 \ldots n$$

Definition

A partition P^* is called a refinement of a partition P, if $P \le P^*$. We also say P^* is finer than P if $P \le P^*$.

If P_1 and P_2 are any two partitions of $[a,b]$, we say that P^* is their common refinement of

$$P^* = P_1 \cup P_2$$

Definition

Let f and g be any two bounded functions on $[a,b]$ and g be monotonic increasing on $[a,b]$.

Corresponding to sub partition

$$P = \{a = x_1 \le x_2 \le \ldots \le x_n = b\}.$$

let

$$\Delta g_i = g(x_i) - g(x_{i-1}), \quad i = 1,2,3 \ldots n$$

///

Clearly, $\Delta g_i \geq 0$. Now we define the same

$$U(f, P, g) = \sum_{i=1}^{n} M_i \Delta g_i \quad (i = 1, 2, 3, \dots n) \text{ --- (i)}$$

$$L(f, P, g) = \sum_{i=1}^{n} m_i \Delta g_i \quad (i = 1, 2, 3 \dots n) \text{ --- (ii)}$$

Where m_i and M_i are the infimum and supremum respectively of f in Δx_i. The sums in (i) and (ii) are respectively called the upper and the lower Darboux sum of f corresponding to the partition.

If m_i and M_i are respectively the lower and the upper bounds of f on $[a, b]$, then we have

$$m \leq m_i \leq M_i \leq M \text{ —— (3)}$$

multiplying through (3) by Δg_i $(i = 1, 2, 3 \dots n)$, gives $m \Delta g_i = m_i \Delta g_i = M_i \Delta g_i \leq M \Delta g_i$ —— (4)

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~~Let f and g adding~~

adding all the terms in (4), yields

$$m \sum_{i=1}^{n} \Delta g_i \leq \sum_{i=1}^{n} m_i \Delta g_i \leq \sum_{n=1}^{n} M_i \Delta g_i \leq M \sum_{i=1}^{n} \Delta g_i$$

$$\Rightarrow m[g(b) - g(a)] \leq L(P, f, g) \leq U(P, f, g) \leq M[f(b) - f(a)]$$

Consider the two integrals

m_i : GLB
M_i = LUB

$$\overline{\int_a^b} f\,dg = \inf U(P,f,g) \quad \text{——— (6)}$$

$$\underline{\int_a^b} f\,dg = \sup L(P,f,g) \quad \text{——— (7)}$$

The Integral in (6) and (7) are respectively called the upper and lower Integrals of f with respect to g over [a, b]

Definition:

The function f is said to be Riemann Steiltjes Integral or simply Integrable ~~over~~ w.r.t g over [a, b] if

$$\overline{\int_a^b} f\,dg = \underline{\int_a^b} f\,dg = \int_a^b f\,dg \quad \text{---- (8)}$$

If (8) holds, then we write

$f \in R_g[a, b]$ or $f \in R_g$

Now from the above Inequalities ~~we~~ have

$$m[g(b)-g(a)] \le L(P,f,g) \le \underline{\int_a^b} f\,dg \le \overline{\int_a^b} f\,dg$$
$$\le U(P,f,g) \le M[g(b)-g(a)]$$

Remark:

=(5) i) The upper and lower Integrals of f w.r.t g over [a, b] always exist for bounded fxns but they may not be equal. If the Integrals exist and are not equal, then such fxn is not Integrable b/c.

ii) Riemann stieltjes Integral of f ~~and g~~ w.r.t g reduces to Riemann Integral If $g(x) = x$

*THEOREM:

If P^* is a refinement of a partition P, then the following holds:

i) $U(P^*, f, g) \leq U(P, f, g)$

ii) $L(P^*, f, g) \geq L(P, f, g)$

PROOF:

Let $P = \{a = x_1 \leq x_2 \leq x_3 - - - \leq x_n = b\}$ be a partition of $[a, b]$. Suppose first that P^* contains just one more point more than P. Let this extra point be c in Δx, i.e

$$y_{n-1} < c < x_i \quad \text{for } (i = 1, \ldots n).$$

Since f is bounded over $[a, b]$, then it is bounded over all the sub-Intervals Δx_i. Let K_1, K_2 and M_i be the upper bound of f in $[x_{i-1}, c]$ $[c, x_i]$ and $[x_{i-1}, x_i]$ respectively.

Clearly, $K_1 \leq M_i$ and $K_2 \leq M_i \; \forall i$

then $U(P^*, f, g) - U(P, f, g)$

$$= \sum_{i=1}^{n} K_i \Delta g_i - \sum_{i=1}^{n} M_i \Delta g_i$$

$$= K_1[g(c)-g(x_{i-1})] + K_2[g(x_i)-g(c)] - M_i[g(x_i)-g(x_{i-1})]$$

$$= (K_1 - M_i)[g(c)-g(x_{i-1})] + (K_2 - M_i)[g(x_i)-g(c)] \leq 0$$

$$\Longrightarrow U(P^*, f, g) \leq U(P, f, g) \qquad \square$$

<u>**THEOREM:**</u> 30/1/23

A function f is Integrable with respect to a function g on $[a,b]$ iff for ~~any~~ every $\varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,g) - L(P,f,g) < \varepsilon.$$

<u>**proof:**</u>

suppose $f \in R_g[a,b]$. This means that

$$\overline{\int_a^b} f\,dg = \underline{\int_a^b} f\,dg = \int_a^b f\,dg \quad \text{———} \quad (1)$$

Let $\epsilon > 0$ be given. Since the ~~approx~~ upper Integral is the Infimum of the set of upper sums, therefore, for the given $\epsilon > 0$, $\exists$ a partition $P_1 = \{x_1, x_2 \ldots x_p\}$ of $[a,b]$ such that

$$U(P_1, f, g) < \overline{\int_a^b} f\,dg + \frac{\epsilon}{2} = \int_a^b f\,dg + \frac{\epsilon}{2} \quad \text{———} \quad (2)$$

Similarly, since the lower Integral is the supremum of the set of lower sums, therefore for the given $\epsilon > 0$, $\exists$ partition $P_2 = \{x_1, x_2, \ldots, x_p\}$

$$L(P_2, f, g) > \underline{\int_a^b} f\,dg - \frac{\epsilon}{2} = \cancel{\int_a^b f\,dg - \frac{\epsilon}{2}} \quad (3)$$

Let $P = P_1 \cup P_2$ be the Common partition of $[a,b]$

Recall that if P^* is a refinement of P, then

$U(P^*, f, g) \le U(P, f, g)$ ——— (4)

$L(P^*, f, g) \ge L(P, f, g)$ ——— (5)

Therefore, we have

$U(P, f, g) \le U(P_1, f, g)$ ——— (6)

and $U(P, f, g) \le U(P_2, f, g)$ ——— (7)

$\therefore$ from (6); we get

$$U(P, f, g) \le U(P_1, f, g)$$
$$< \int_a^b f\,dg + \epsilon/2$$
$$< L(P_2, f, g) + \epsilon \quad \left\{\begin{array}{l}\text{by adding } \epsilon \text{ to both} \\ \text{sides of (3)}\end{array}\right\}$$
$$\le L(P, f, g) + \epsilon, \text{ using (5)}$$

i.e

$$U(P, f, g) < L(P, f, g) + \epsilon$$

$$\Rightarrow U(P, f, g) - L(P, f, g) < \epsilon$$

($\Leftarrow$) Let $\epsilon > 0$ be given and P be a partition of $[a, b]$ for which

$$U(P, f, g) - L(P, f, g) < \epsilon$$

for any partition P, we know that

$$L(P, f, g) \le \underline{\int_a^b} f\,dg \le \overline{\int_a^b} f\,dg \le U(P, f, g)$$

$$\Rightarrow \overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg \le U(P, f, g) - L(P, f, g) < \epsilon.$$

i.e $\overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg < \epsilon, \ \forall \epsilon > 0$

But a non-negative number can be less than every positive number if it is given. Hence,

$$\overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg = 0 \Rightarrow \overline{\int_a^b} f\,dg = \underline{\int_a^b} f\,dg.$$

This proves that $f \in R_g[a,b]$.

Definition Corresponding to a partition P of $[a,b]$ and $t_i \in \Delta x_i$, consider the sum

$$S(P,f,g) = \sum_{i=1}^{n} f(t_i)\Delta g_i$$

We say that the $S(P,f,g)$ converges to L as $\mu(P) \to 0$, i.e.

$\lim S(P,f,g) = L$, if for every $\varepsilon > 0$, there exists
$\mu(P) \to 0$

a $\delta > 0$ such that

$|S(P,f,g) - L| < \varepsilon$, for every partition

$P = \{a = x_1 \le x_2 \le \dots \le x_n = b\}$ of $[a,b]$ with non mesh $\mu(P) < \delta$ and for every choice $t_i \in \Delta x_i$ $(i = 1,2,\dots n)$.

THEOREM:

If $\lim S(P,f,g)$ exists as $\mu(P) \to 0$, then $f \in R_g[a,b]$ and $\lim\limits_{\mu(P)\to 0} S(P,f,g) = \int_a^b f\,dg$.

proof:

suppose that $\lim\limits_{\mu(P)\to 0} S(P,f,g) = L$. This means that for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that for every partition P of $[a,b]$ with the mesh $\mu(P) < \delta$ and every choice $t_i \in \Delta x_i$ we get

$$\left| S(P,f,g) - L \right| < \epsilon/2$$

$$\Rightarrow -\epsilon/2 < S(P,f,g) - L < \epsilon/2$$

$$\Rightarrow L - \epsilon/2 < S(P,f,g) < L + \epsilon/2 \quad \text{——— (1)}$$

Taking the infimum and supremum of the sums $S(P,f,g)$ equation (1) becomes

$$L - \epsilon/2 < L(P,f,g) \leq U(P,f,g) < L + \epsilon/2$$

$$\Rightarrow U(P,f,g) - L(P,f,g) < (L + \epsilon/2) - (L - \epsilon/2) = \epsilon$$

i.e. $U(P,f,g) - L(P,f,g) < \epsilon$ ——— (2)

$$\Rightarrow f \in R_g[a,b].$$

Now, since $S(P,f,g)$ and $\int_a^b f\,dg$ both lie between $U(P,f,g)$ and $L(P,f,g)$, then we have

$$\left| S(P,f,g) - \int_a^b f\,dg \right| \leq U(P,f,g) - L(P,f,g) < \epsilon, \text{ using (2)}$$

$$\Rightarrow \lim_{\mu(P)\to 0} S(P,f,g) = \int_a^b f\,dg \qquad \square$$

<u>Remark</u>

The above theorem, asserts that the existence of the limit of $S(P,f,g)$ implies that $f \in R_g[a,b]$. The existence of the limit is a sufficient condition for $f \in R_g[a,b]$ but it is not a necessary condition that is, functions exist which are integrable but for which the limit of $S(P,f,g)$ do not exist.

Theorem: If f is Continuous on $[a,b]$, then $f \in R_g[a,b]$.

proof (exercise)

THEOREM: If f is monotonic on $[a,b]$ and g is Continuous on $[a,b]$, then $f \in R_g[a,b]$

Quiz Let P^* be a refinement of the partition P. Then show that the following Inequality is valid.

$$L(P, f, g) \leq L(P^*, f, g).$$

Example 2/2/23

A function g Increases on $[a,b]$ and is Continuous at k, where $a \leq k \leq b$. Another function f is such that

$f(k) = 1$ and $f(x) = 0$, for $x \neq k$.

prove that $f \in R_g[a,b]$ and $\int_a^b f\, dg = 0$.

Soln

Let $P = \{a = x_0 \leq x_1 \leq \ldots \leq x_n = b\}$ be a partition of $[a,b]$ and let $k \in \Delta x_i$, $x_{i-1} \leq x \leq x_i$

By Continuity of g on $[a,b]$, for $\epsilon > 0$, we need to find a $\delta > 0$ such that

$$|g(x) - g(k)| < \epsilon/2, \text{ for } |x - k| < \delta$$

Again since g is Increasing on $[a,b]$ we get $g(x) - g(k) < \epsilon/2$, if $0 < |x - k| < \delta$

$$g(x) - g(x) < \epsilon/2, \text{ if } 0 < |x - x| < \delta.$$

Let P be a partition of $[a,b]$ with $\mu(P) < \delta$

then, $\Delta g_i = g(x_i) - g(x_{i-1})$

$$= g(x_i) - g(k) + g(x) - g(x_{i-1})$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence,

$$S(P, f, g) = \sum_{i=1} f(t_i) \Delta g_i = \begin{cases} \Delta g_i, & t_i = k \\ 0 i, & t_i \neq k. \end{cases}$$

In either case,

$$\lim_{\mu(P) \to 0} S(P, f, g) = 0 < \infty$$

$\to f \in R_g[a,b]$. Hence, $\lim_{\mu(P) \to 0} S(P, f, g) = \int_a^b f dg = 0$

Exercise

If f is a function bounded on $[-1,1]$, and g_1, g_2, g_3 are three functions defined as follows

$$g_1(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0, \end{cases}$$

$$g_2(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$g_3(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$$

prove that $f \in R_g[-1,1]$. If f is continous at $x = 0$ and then $\int_{-1}^{1} f dg = f(0)$.